

A method for evaluating models that use galaxy rotation curves to derive the density profiles

Álefe O. F. de Almeida, Oliver F. Piattella and Davi C. Rodrigues^{*}

Departamento de Física, Universidade Federal do Espírito Santo, Av. Fernando Ferrari 514, Vitória, ES, 29075-910 Brazil

ABSTRACT

There are some approaches, either based on General Relativity (GR) or modified gravity, that use galaxy rotation curves to derive the matter density of the corresponding galaxy, and this procedure would either indicate a partial or a complete elimination of dark matter in galaxies. Here we review these approaches, clarify the difficulties on this inverted procedure, present a method for evaluating them, and use it to test two specific approaches that are based on GR: the Cooperstock-Tieu (CT) and the Balasin-Grumiller (BG) approaches. Using this new method, we find that neither of the tested approaches can satisfactorily fit the observational data without dark matter. The CT approach results can be significantly improved if some dark matter is considered, while for the BG approach no usual dark matter halo can improve its results.

Key words: gravitation, dark matter, galaxies: spiral, galaxies: kinematics and dynamics

1 INTRODUCTION

Besides the unknown nature of dark matter, the standard model of cosmology (Λ CDM) is also facing difficulties (e.g., Donato et al. 2009; de Blok 2010; Oh et al. 2011; Boylan-Kolchin et al. 2011, 2012; Weinberg et al. 2013; Pawłowski et al. 2015). There is hope that these issues may be solvable within the Λ CDM model (e.g., Governato et al. 2012; Del Popolo et al. 2014; Oñorbe et al. 2015), but the solutions depend on baryonic physics details with which is difficult to deal semi-analytically or through simulations. On the other hand, the answer may be related with more issues than the baryonic physics alone, and may depend on the nature of dark matter (Moore 1994; Colin et al. 2000; Hu et al. 2000; Zavala et al. 2009; Foot & Vagnozzi 2015), on refinements on the gravitational side (Capozziello & De Laurentis 2012; Lora et al. 2013; Rodrigues et al. 2014), or perhaps on both.

Galaxy rotation curves (RCs) constitute one of the most clear and useful test on the existence of either dark matter or non-Newtonian gravity in galaxies at low redshift. The determination of the dark matter profile in a galaxy is based on the following schematic procedure (see e.g., Sofue & Rubin 2001; Courteau et al. 2014): the observed light is converted into mass densities for the stellar and the gaseous parts. For the stellar part, the conversion depends on the stellar mass-to-light ratio, which depends on the dominant stellar population. From these mass densities, one derives the corresponding Newtonian potentials, and therefore their in-

dividual contributions to the RC. These contributions are typically far from being sufficient to reproduce the observed RC, the difference being attributed to dark matter. In order to compute the dark matter contribution, the usual procedure is to assume a dark matter halo profile that depends on some free parameters which are fitted to the observed RC.

From Newtonian gravity without assumptions on the matter distribution, it is not possible to infer the mass density of a disk galaxy from the observational RC alone, even if it is assumed that all the matter is in a thin axisymmetric disk (Binney & Tremaine 1988). What can be done is to compare the RC generated by a given mass density profile, with some free parameters, to the observed RC.

On the other hand, some non-Newtonian proposals (Cooperstock & Tieu 2007; Coimbra-Araujo & Letelier 2007; Dey et al. 2015; Magalhaes & Cooperstock 2015) use the inverse procedure: the observed RC is used as the input from which the mass distribution is derived. For some galaxies, the RC fits of these theories seem satisfactory, but these publications lack a detailed investigation with respect to the baryonic matter data inferred from observations. Models that use this inverse route have not been yet properly tested and confronted with results from other approaches, it is the purpose of this new method to be able to properly test and compare them. Also, there are other nontrivial approaches that have never been tested with RC data, and this inverse procedure may prove useful to test them (e.g., Vogt & Letelier 2007; Balasin & Grumiller 2008; Rahaman et al. 2008; Vieira & Letelier 2014).

To address the latter issue, we propose here the effec-

^{*} E-mail: davi.rodrigues@cosmo-ufes.org

tive Newtonian RC method. To exemplify it, two relativistic approaches are selected, the one of the Refs. (Cooperstock & Tieu 2007, 2008; Carrick & Cooperstock 2012), which we label CT, and the one of Ref. (Balasin & Grumiller 2008), which we label BG. Both of them use GR in 4D spacetime as the source for gravitational dynamics. Details, merits and criticisms on these approaches are presented in the next sections. This is the first time that the BG approach is studied with realistic galaxy data.

The next section reviews the CT and the BG approaches, sec. 3 presents the effective Newtonian method in generality and its two applications, sec. 4 shows the results, and in sec. 5 we present our conclusions and discussions.

2 A BRIEF REVIEW ON TWO RELATIVISTIC APPROACHES

2.1 General considerations

In order to determine the distribution of dark matter in galaxies, the standard approach is to use Newtonian gravity. The motivation for doing so comes usually from the following: *i*) the assumption that GR is the gravitational theory to be considered; *ii*) that galaxies seem to be stationary systems whose Newtonian potential is small (typically about $\sim 10^{-6} - 10^{-8}$, in $c = 1$ units), and *iii*) that the typical speeds are at most about a few hundred km/s (i.e., $\lesssim 10^{-3}$, using $c = 1$ units). These small numbers *suggest* that GR corrections to Newtonian dynamics, namely on the rotation curve (RC), are smaller than 1%, and therefore significantly smaller than the typical uncertainties associated with the astrophysical data from galaxies. Some authors agree with the assumption of using GR in galaxies (item *i*), but found that those small numbers in the items *ii* and *iii* may lead to significant consequences on dark matter distribution in galaxies and corrections to the RCs larger than 10% (Cooperstock & Tieu 2007; Carrick & Cooperstock 2012; Magalhaes & Cooperstock 2015; Balasin & Grumiller 2008; Ramos-Caro et al. 2012).

The CT approach (Cooperstock & Tieu 2006, 2007, 2008; Carrick & Cooperstock 2012; Magalhaes & Cooperstock 2015) has received a number of criticisms on the theoretical basis, which the authors claim to have answered (Carrick & Cooperstock 2012). These criticisms focus on whether the CT approach is indeed fully embedded in GR with a single kind of matter given by a disk of dust. Apart from the theoretical issues, the authors in Refs. (Cooperstock & Tieu 2007; Carrick & Cooperstock 2012) also do a strong claim, which is that their approach is capable of reproducing the internal dynamics of about 10 commonly studied galaxies without the need for dark matter, or with only a small amount of it. It appears that, apart from the present work, there is only a single work that criticises the CT phenomenological consequences (Fuchs & Phleps 2006). Their criticism considers a single galaxy, the Milky Way, and not its rotation curve, but only the velocity dispersion of stars outside the galactic plane or in the Sun neighbourhood. The latter is a valid criticism, but too specific and more prone to observational systematical errors that could erroneously invalidate the model.

The BG approach (Balasin & Grumiller 2008) is actually a bifurcation of the CT approach that avoids certain issues on the CT solution at the galactic plane ($z = 0$) (Vogt & Letelier 2005; Balasin & Grumiller 2008). It is claimed that this approach cannot remove dark matter in galaxies, but it can significantly reduce its total amount (Balasin & Grumiller 2008).

2.2 The Cooperstock & Tieu (CT) approach

The CT approach (Cooperstock & Tieu 2007, 2006, 2008; Carrick & Cooperstock 2012; Magalhaes & Cooperstock 2015) starts from the assumption that all the relevant matter in a galaxy can be modelled by an axisymmetric stationary dust fluid, and that the spacetime metric can be written as, using $c = 1$ units and standard conventions of the cylindrical coordinates (r, ϕ, z) ,

$$ds^2 = -e^w(dt - Nd\phi)^2 + e^{-w}r^2d\phi^2 + e^{\nu-w}(dr^2 + u dz^2), \quad (1)$$

where w, N, ν, u are functions that only depend on the coordinates r, z . The above line element is not necessarily the most convenient to work on the dynamics of galaxies, but it is the most general with the desired symmetry (e.g., see Chap. 7 of Wald 1984). Since all the considered matter is dust, it is possible to select coordinates to reduce the number of functions that the metric depends on (Balasin & Grumiller 2008), such that the line element becomes

$$ds^2 = -(dt - Nd\phi)^2 + r^2d\phi^2 + e^\nu(dr^2 + dz^2). \quad (2)$$

By either performing a suitable coordinate change (Carrick & Cooperstock 2012) or an ADM splitting to unveil the lapse function and the shift vector (Balasin & Grumiller 2008), an asymptotic observer at rest with respect to the galaxy center would perceive space rotation with a velocity profile given by

$$V = \frac{N}{r}. \quad (3)$$

The GR field equations impose limits on the form of N , in particular from the field equations,

$$N_{rr} + N_{zz} - \frac{N_r}{r} = 0. \quad (4)$$

For $z \geq 0$, the following is a valid expression for V (Cooperstock & Tieu 2007),

$$V_{CT}(r, z \geq 0) = - \sum_n D_n e^{-k_n z} J_1(k_n r), \quad (5)$$

where J_α is a Bessel function of the first kind, and k_n and D_n are arbitrary constants.

The main point of the CT approach is to present a non-Newtonian solution for galaxies that can fit the data without dark matter or a small amount of it, and this non-Newtonian solution needs not to be the most general case. To this end, Cooperstock & Tieu (2007) have found that they could achieve interesting results by setting the constants k_n to be the n -th root of the Bessel function $J_0(k_n r_{\max})$, where r_{\max} is the radius of the farthest observed circular velocity data of a given galaxy. Therefore, at $z = 0$, it is not a surprise that eq. (5) can fit very well the rotation curve of galaxies, it is just a kind of Fourier-Bessel series, which can actually fit almost any curve defined in the interval $(0, r_{\max})$.

The nontrivial part of eq. (5) is the z dependence. Hence, it should be clear that the fact that this approach can match very well the observational rotation curve data at $z = 0$ is irrelevant for the phenomenology, it is a triviality, since it can fit any curve. What is not trivial is whether the corresponding inferred mass distribution matches the observed baryonic density.

The extension of eq. (5) by using $V(r, z) = V(r, -z)$ is a problematic one and was criticised in particular by Vogt & Letelier (2005); Balasin & Grumiller (2008) (see however Carrick & Cooperstock 2012).

The connection between the velocity profile and the matter distribution is derived from the following GR equation,

$$\frac{N_r^2 + N_z^2}{r^2} = 8\pi G\rho e^\nu \approx 8\pi G\rho. \quad (6)$$

To apply this approach to galaxy RC's, from the RC data one uses eq. (5) to derive the D_n coefficients. The fit of the curve (5) to the observational RC can be as good as one wants, the higher precision one demands, the larger is the number of D_n 's to be fitted. Typically this approach uses about 10 coefficients (Cooperstock & Tieu 2007) (this is just a matter of convention and was found to be suitable to a certain number of galaxies). Hence, the RC fit alone of this approach is physically irrelevant. The physically important consequences are derived from the matter distribution ρ that is inferred from the RC from eq. (6).

The high number of free parameters that the CT approach uses, when confronted to phenomenological results (e.g., Salucci et al. 2007), seem to indicate alone that either this model is unrealistic or that there must exist some way to reduce its number of free parameters. Indeed, the rule to only pick the first n roots of the Bessel function is arbitrary. It may exist a rule to select three particular roots of the Bessel function depending on certain galaxy parameters (e.g., the disc scale length), and hence three D_n constants only, which would lead to reasonable galaxy RC fits. Hence, although it is inconvenient that no such rule for selecting the best three D_n parameters is known, the argument on the number of free parameters is not sufficient to discredit this approach.

2.3 The Balasin & Grumiller (BG) approach

The BG approach is a bifurcation of the CT one. It starts from the same line element (2) and the same energy momentum tensor, but their solution respects the reflection symmetry about the $z = 0$ plane, contrary to eq. (5).

The exact GR field equations derived from the line element (2) and the energy momentum tensor $T_\mu^\nu = \rho U_\mu U^\nu$ read

$$2r\nu_r + N_r^2 - N_z^2 = 0, \quad (7)$$

$$r\nu_z + N_r N_z = 0, \quad (8)$$

$$\nu_{rr} + \nu_{zz} + \frac{1}{2r^2}(N_r^2 + N_z^2) = 0, \quad (9)$$

$$N_{rr} + N_{zz} - \frac{N_r}{r} = 0, \quad (10)$$

$$\frac{N_r^2 + N_z^2}{r^2} = 8\pi G\rho e^\nu. \quad (11)$$

Balasin & Grumiller (2008) present the following solution for eq. (10),

$$N(r, z) = A_0 + \int_0^\infty \cos(\lambda z)(r\lambda)A(\lambda)K_1(\lambda r)d\lambda, \quad (12)$$

where $A(\lambda)$ is a “sufficiently regular” arbitrary function, A_0 is a constant and K_1 is a modified Bessel function of the second kind. Since the relation between V and N is given by eq. (3), it should be clear that selecting $A(\lambda)$ to fit the observed RC should not be seen as the physical output of this approach, but the physical input. It is shown that a suitable choice of the $A(\lambda)$ function can lead to the following velocity profile,

$$V_{\text{BG}}(r, z) = \frac{(R - r_0)V_0}{r} + \frac{V_0}{2r} \sum_{\pm} \left(\sqrt{(z \pm r_0)^2 + r^2} - \sqrt{(z \pm R)^2 + r^2} \right), \quad (13)$$

with, $|z| < r_0$. Hence, at $z = 0$,

$$V_{\text{BG}}(r, 0) = \frac{V_0}{r} \left(R - r_0 + \sqrt{r_0^2 + r^2} - \sqrt{R^2 + r^2} \right). \quad (14)$$

This profile includes three stages, first the linear increase (for $r \lesssim r_0$), then the constant velocity $V \sim V_0$ regime (for $r_0 \lesssim r \lesssim R$), and a $1/r$ decrease for $r \gg R$. In practice, for many galaxies, the parameter R cannot be accurately derived from the observational RC, since the transition to a decreasing RC cannot be seen up to the last RC data.

For circular velocities much lower than the speed of light, and assuming that at $r < r_0$ this approach should coincide with Newtonian gravity, Ref. (Balasin & Grumiller 2008) shows that ν is (close to) a constant. Finally, by comparing the differences between Newtonian gravity and their GR approach at the plateau part of the RC, it is argued that this GR approach may significantly reduce the need of dark matter (the estimated differences being about 30% of the total matter).

In the following, to simplify the problem in this first step, we will consider both the CT and the BG approaches without dark matter.

3 THE EFFECTIVE NEWTONIAN ROTATION CURVE METHOD

3.1 General considerations

The purpose of this method is to properly and feasibly evaluate models in which the observational RC is used as the model input, while the mass density profile is derived from the latter. The essential feature is to circumvent the use of the commonly unknown matter density error bars by a proper, model dependent, transposition of the observational RC error bars to an effective Newtonian RC. In the end, the method provides an effective RC with error bars that should be fitted with the usual Newtonian procedures.

For the majority of the works on galaxy RC data, and for diverse reasons, there is no profile stating the values of the baryonic density at each radius and its corresponding uncertainty. Therefore, if a model can derive a baryonic density profile by certain means, it is not obvious how to compare it with the expected baryonic profile from the observations.

The proposed method uses that the relevant uncertainties are already in part encoded in the RC error bars. In Refs. (de Blok et al. 2008; Gentile et al. 2004), like in many others, changes in the redshift data at the same galaxy radius are the main contribution to the RC error bars at that radius. Therefore, the RC error bars contain information on the violation of perfect axial symmetry, and hence they include information on the maximal confidence one can have on any axially symmetric model.

The method that is here proposed depends on the realisation of two minimization procedures. The first one is to derive the model parameters that best fit the observed RC (this fit does not depend on neither the baryonic or the dark matter densities). The second minimization is used to derive the baryonic (and dark matter) parameters, and to yield the relevant quantities to evaluate the goodness of the fit. If one is suspicious on the value of a baryonic model parameter, say the disc scale length, or wants to consider it within a certain range, one can do the last fit with different values of that parameter (without the need of redoing the first fit). The method can be briefly summarised in the following steps:

- The model circular velocity at $z = 0$, which is designated by $V(r, p_i)$, where p_i represent the model parameters, is fitted to the observed RC. This RC is described by the table whose k -th line reads $(r_k, V_{\text{obs},k}, \delta V_{\text{obs},k})$, where r_k is the radius of the galaxy whose corresponding circular velocity is $V_{\text{obs},k}$ with a 1σ error bar given by $\delta V_{\text{obs},k}$. The fit determines the best fit parameters \bar{p}_i and the corresponding error bars δp_i .

- From $V(r, \bar{p}_i \pm \delta p_i)$ one can (numerically) determine the corresponding mass density profile as a function of the model parameters, $\rho(r, z, \bar{p}_i \pm \delta p_i)$, for instance from eqs. (3, 6).

- From $\rho(r, z, \bar{p}_i \pm \delta p_i)$ one can determine the effective Newtonian circular velocity at $z = 0$. The latter is written as $V_{\text{eN}}(r, \bar{p}_i \pm \delta p_i)$ and it is defined as being the circular velocity derived from Newtonian gravity for the matter density $\rho(r, z, \bar{p}_i \pm \delta p_i)$.

- The effective Newtonian RC data with error bars is built from V_{eN} . These data can be expressed as a table whose k -th line is given by $(r_k, \bar{V}_{\text{eN},k}, \delta V_{\text{eN},k})$, where r_k assumes the same values of the original data on the observational RC, $\bar{V}_{\text{eN},k} = V_{\text{eN}}(r_k, \bar{p}_i)$ and $\delta V_{\text{eN},k}$ is an approximation for the corresponding 1σ error bar, which is detailed afterwards.

- The astrophysical expectation on the gas and stellar densities, together possibly with a given dark matter profile, are used to derive the Newtonian circular velocity V_{N} , which will depend on baryonic parameters (like the mass-to-light ratios) and possibly on dark matter parameters as well.

- If the gravitation theory being considered is compatible with both the observational RC and the matter content assumed for the galaxy, then V_{eN} and V_{N} should be mutually compatible. Hence, one fits V_{N} to the effective Newtonian RC data, thus deriving the baryonic (and dark matter) parameters, and deriving the quantities χ^2 and χ_{red}^2 . The latter are the quantities that have physical information on the quality of the fit and that can be compared to other approaches.

3.2 The fit procedure step by step

Here we describe in detail the procedures associated to the proposed method in four steps:

(i) *The derivation of \bar{p}_i and ρ .* The minimisation of certain χ^2 is used to compute the best fit parameters for $V(r, p_i)$ in regard to the observed RC data. The corresponding χ^2 quantity is

$$\chi_p^2 = \sum_{k=1}^N \left(\frac{V(r_k, p_i) - V_{\text{obs},k}}{\delta V_{\text{obs},k}} \right)^2. \quad (15)$$

The subscript p is a reminder that the purpose of χ_p^2 is to derive the model parameters p_i from the observational RC data (this does not constitute the main model results). The values of p_i that minimise χ_p^2 are denoted by \bar{p}_i , and N is the total number of observational data points of the circular velocity V_{obs} . From the knowledge of $V(r, \bar{p}_i)$ it is straightforward to evaluate the matter density $\rho(r, z, \bar{p}_i)$. It is also possible to evaluate ρ for all the values of p_i inside the range given by the error bars δp_i , hence one can derive $\rho(r, z, \bar{p}_i \pm \delta p_i)$.

(ii) *The derivation of V_{eN} .* The effective Newtonian circular velocity can be derived by solving the Poisson equation, $\nabla^2 \Phi(r, z, \bar{p}_i) = 4\pi G \rho(r, z, \bar{p}_i)$, and using $V_{\text{eN}}^2(r, \bar{p}_i) = r \partial_r \Phi(r, \bar{p}_i)$. In particular, the expression for V_{eN} can be directly evaluated from (Binney & Tremaine 1988)

$$\begin{aligned} V_{\text{eN}}^2(r, p_i) &= r \partial_r \Phi(r, z=0, p_i), \\ &= -Gr \partial_r \int_{-\pi}^{\pi} d\varphi' \int_{-\infty}^{\infty} dz' \int_0^{\infty} dr' \times \\ &\quad \times \frac{\rho(r', z', p_i)}{\sqrt{r^2 + r'^2 + z'^2 - 2rr' \cos(\varphi')}} r', \\ &= -2Gr \int_0^{\infty} dz' \int_0^{\infty} dr' \times \\ &\quad \times \rho(r', z', p_i) \partial_r \left(\frac{4K \left(\frac{4rr'}{(r+r')^2 + z'^2} \right)}{\sqrt{(r+r')^2 + z'^2}} \right) r'. \end{aligned} \quad (16)$$

In the above, K is the complete elliptic integral defined by

$$K(x) = F(\pi/2, x) = \int_0^{\pi/2} d\theta (1 - x \sin^2(\theta))^{-1/2}. \quad (17)$$

From the above, it is possible to derive V_{eN} for all the values of p_i inside their 1σ uncertainties, that is, one can find $V_{\text{eN}}(r, \bar{p}_i \pm \delta p_i)$.

(iii) *The effective Newtonian RC.* The purpose of this step is to generate the relevant data, with error bars, that should be used in the next and final fitting procedure. The observational RC is described by the data $(r_k, V_{\text{obs},k}, \delta V_{\text{obs},k})$, where k runs from 1 to N . The effective Newtonian RC data are given by $(r_k, \bar{V}_{\text{eN},k}, \delta V_{\text{eN},k})$. In order to avoid the introduction of any bias towards any radii, the same radial values r_k used for the observational RC also appear for the effective Newtonian RC. The quantity $\bar{V}_{\text{eN},k}$ is simply $V_{\text{eN}}(r_k, \bar{p}_i)$ and $\delta V_{\text{eN},k}$ is its corresponding 1σ error bar. A straightforward procedure to derive the latter goes as follows: firstly one finds $V_{\text{max},k}$ and $V_{\text{min},k}$, which are respectively the maximum and the minimum of $V_{\text{eN}}(r_k, p_i)$, with fixed r_k , such

that $\chi^2(p_i) \leq \chi_{\min}^2 + \Delta\chi^2$, where $\Delta\chi^2$ is the constant associated to a 1σ uncertainty considering the total number of the model parameters (p_i). This guarantees that $V_{\max,k}$ is the maximum value achievable for $V_{eN,k}$ inside the 1σ confidence region. Ideally one should compute the full probability density function (PDF), but depending on the model it may be either exactly valid or be a reasonable approximation to assume a Gaussian distribution. If the error bars are not exactly symmetric (but are not far from being symmetric), the 1σ uncertainty $\delta V_{eN,k}$ is set as the maximum between $V_{\max,k} - \bar{V}_{eN,k}$ and $\bar{V}_{eN,k} - V_{\min,k}$. It should be noted that $\Delta\chi^2$ increases with the number of model parameters p_i , and hence in general the larger is the number of parameters p_i , the larger will be the uncertainties $\delta V_{eN,k}$.

(iv) *The derivation of the baryonic and dark matter parameters.* Since all the galaxy matter is composed by either baryonic or dark matter, the total Newtonian circular velocity can be expressed by

$$V_N^2 = V_{\text{disk}}^2 + V_{\text{bulge}}^2 + V_{\text{gas}}^2 + V_{\text{dark matter}}^2. \quad (18)$$

To be clear, V_N^2 is directly derived from certain matter densities as given above, while V_{eN}^2 is derived from the observational RC and from the use of the chosen non-Newtonian gravitation.

For concreteness, here it is considered that the stellar mass-to-light ratios of the bulge and the disk (Υ_{*B} and Υ_{*D}) are the only baryonic parameters that are not sufficiently constrained by the observations and need to be fitted. The dark matter contribution will not be considered at the moment, that is, $V_{\text{dark matter}}^2 = 0$. In conclusion, for the assumptions above, $V_N^2 = V_N^2(r, \Upsilon_{*B}, \Upsilon_{*D})$.

If the gravitation theory being considered is compatible with the observational RC and the matter content assumed for the galaxy, then V_{eN} and V_N should be mutually compatible. Since V_N depends on free parameters, one should evaluate a second and last χ^2 minimization, whose quantity to be minimized reads,

$$\chi^2 = \sum_{k=1}^N \left(\frac{V_N(r_k, \Upsilon_{*D}, \Upsilon_{*B}) - \bar{V}_{eN,k}}{\delta V_{eN,k}} \right)^2. \quad (19)$$

It is this last χ^2 , and the reduced chi-square computed from it (χ_{red}^2), the quantities that should be used to compare different approaches, not χ_p^2 .

3.3 Application to the CT approach

To apply the effective Newtonian method to the CT approach, we follow the steps detailed in Sec. 3.2.

(i) *The derivation of \bar{p}_i and ρ .* The results for \bar{p}_i and its corresponding error bars can be seen in Table 2. The p_i parameters for this approach correspond to the D_n parameters in eq. (5).

For all the six galaxies of this sample we followed the procedure of Cooperstock & Tieu (2007) of adopting 10 parameters to be fit. For this first fit, the CT approach with 10 parameters could easily fit the observational RC. This can be seen from the values of $\chi_{p,\text{red}}^2$ in Table 3. Although 10 parameters is more than the usual number of parameters used to fit galaxies, this quantity depends on the chosen profile. For the CT approach, which uses eq. (5), less than five

parameters only leads to good fits (i.e., $\chi_{p,\text{red}}^2 \sim 1$) for a few galaxies, typically those whose RC slowly and smoothly increases and hence do not need the high frequency terms of the expansion (5). There are examples in which 10 parameters are not sufficient (Magalhaes & Cooperstock 2015).

The derivation of ρ from the fitted circular velocity V , at the region with observational RC data, comes from the combination of eqs. (3, 5, 6).

In general, for the evaluation of V_{eN} , it is necessary to consider an extension of ρ beyond the farthest observational RC data, whose radius is r_{\max} . Namely, the larger is the density beyond r_{\max} , the smaller becomes V_{eN} close to r_{\max} (this is a known Newtonian effect in axisymmetric systems Binney & Tremaine 1988). In principle, one can extend ρ beyond r_{\max} by simply extending the circular velocity curve towards larger r and using eq. (5). But, as explained in detail by Cooperstock & Tieu (2006), there is no need to use the same D_n and the same k_n beyond r_{\max} , and physically reasonable extensions usually require different values for the latter parameters. From the phenomenological perspective, for sure the baryonic mass density of galaxies must drop at larger radius. As a phenomenologically simple and viable approximation for the total baryonic matter beyond the last observed RC data, we adopt

$$\rho(r \geq r_{\max}, z) = e^{(r_{\max}-r)/r_d} \rho(r_{\max}, z). \quad (20)$$

This extension is specially natural for the case of a disk galaxy with negligible gas content, since it is just an extension of a Freeman disk (Freeman 1970). The gas density usually decays slower than the stellar component, hence for galaxies with significant amount of gas, the above approximation will cease to be a good one at some radius. Nonetheless, the impact of such deviations on V_{eN} is insignificant, since only the density beyond but close to r_{\max} should contribute significantly to V_{eN} . Moreover, due to the exponential decrease, and the small density at r_{\max} , changes on r_d by a factor of two have small or negligible impact on V_{eN} .

(ii) *The derivation of V_{eN} .* Since, with the extension above, ρ is known in the complete space, deriving V_{eN} reduces to computing the integral (16). A technical difficulty can be promptly spotted, and it comes from the large number of parameters that $V_{eN}(r, D_n)$ depends on. This difficulty will have consequences to the next step. On the other hand, it is computationally easy to derive the effective Newtonian circular velocity with the best fit D_n parameters, which is written as $\bar{V}_{eN}(r)$.

(iii) *The effective Newtonian RC data.* As detailed in the previous section, these data can be expressed through a table given by $(r_k, \bar{V}_{eN,k}, \delta V_{eN,k})$, hence at this step one should compute the error bars $\delta V_{eN,k}$. To this end, it is necessary to perform both a minimization and a maximization of $V_{eN}(r, D_n)$ with the constraint $\chi^2(p_i) \leq \chi_{\min}^2 + \Delta\chi^2$ at each radius r_k . Thus, for each observational RC data point, and for each one of the six galaxies, one should derive constrained maximizations and minimizations with 10 free parameters.

For the particular case of the CT approach, it is not essential to compute $\delta V_{eN,k}$ to conclude that this model (without a large amount of dark matter) cannot describe the astrophysical data of galaxies, since in spite of the error bars val-

ues, the V_{eN} and V_N are systematically incompatible. Moreover, it is not computationally easy to evaluate $\delta V_{eN,k}$ for the CT approach with ten D_n parameters.

To evaluate the CT approach with fewer than 10 parameters is helpful as an illustration and to serve as a basis for an estimation of δV_{eN} when the 10 parameters are considered. The galaxy ESO 116-G12 was selected to be analysed with the CT approach and with only three D_n parameters. The results are in Fig. 1 and Table 1. The derived values of δV_{eN} ranges from 0.6 km/s to 4.5 km/s. With the exception of the first point, all the others have δV_{eN} larger than 1 km/s. The mean δV_{eN} is about 3 km/s.

(iv) *The derivation of the baryonic parameters.* From the fit of V_N to the effective Newtonian RC, one derives χ^2 , χ^2_{red} , Υ_{*B} and Υ_{*D} . This is a straightforward procedure, and the results are presented and commented in the next section.

3.4 Application to the BG approach

It is easier to apply the effective Newtonian method to the BG than to the CT one for some reasons. The numerical integrals are faster to compute, the model always use 3 parameters (p_i) instead of 10, and the extension of ρ beyond the last observed RC data, r_{max} , is already included as part of this model.

(i) *The derivation of \bar{p}_i and ρ .* The results for \bar{p}_i and its corresponding error bars are displayed in Table 2. The p_i parameters for this approach correspond to the three parameters in eq. (14), i.e., R , r_0 and V_0 . This first fit that fixes the p_i parameters yields values for $\chi^2_{p,red}$ in the range from 0.5 to 1.6, thus indicating that the velocity profile of this approach is reasonable for describing the RC of galaxies.

The derivation of ρ from the fitted circular velocity V comes from the combination of eqs. (3, 11, 14). The extension of ρ beyond r_{max} is direct in this BG approach, and essentially it depends on a single parameter (R). For some galaxies the value of this parameter can be constrained to lie within some kpc's, but for others, specially those whose RC is monotonously increasing up to r_{max} , there is no maximum for R .

(ii) *The derivation of V_{eN} .* The effective Newtonian circular velocity V_{eN} is directly computed from eq. (16).

(iii) *The effective Newtonian RC data.* At this step one should compute the error bars $\delta V_{eN,k}$. To this end, it is necessary to perform both a minimisation and a maximisation of $V_{eN}(r, r_0, V_0, R)$ with the constraint $\chi^2(r_0, V_0, R) \leq \chi^2_{min} + \Delta\chi^2$ at each radius r_k . Thus, for each observational RC data point, and for each one of the six galaxies, one should derive constrained maximisations and minimisations with 3 free parameters. The derived error bars were either symmetric or close to symmetric, and they were all symmetrized taking the largest value. This was done for all the six galaxies that are in this work evaluated.

(iv) *The derivation of the baryonic parameters.* From the fit of V_N to the effective Newtonian RC, one derives χ^2 , χ^2_{red} , Υ_{*B} and Υ_{*D} . This is a straightforward procedure, and the results are presented and commented in the next section.

4 RESULTS

The results of the fit procedures for CT and BG are in the Tables 2, 3 and 4, and the RC plots are shown in Fig. 2.

The CT approach with 10 parameters needs at least about 10^5 times more computational time than the BG approach, hence some approximation for δV_{eN} was necessary. All the error bars on this approach were taken to be the same with a common value of 4.5 km/s, based on the maximum error value of the case with 3 parameters presented in Fig. 1. The latter is plausible since additional parameters on this approach only add Bessel functions of higher frequency, and hence the error bars derived with ten free parameters are not expected to be much larger than the three parameters ones.

The plots in Fig. (2) clearly show that, for all the galaxies modelled with the CT approach, the form of the effective Newtonian RC (the grey squares in the plots) systematically does not match the form of the Newtonian circular velocity V_N . For all these six galaxies, the best fit Newtonian circular velocity V_N is too high for small radii, and becomes too low at large radii. The latter behaviour is a clear indication that adding a dark matter halo would significantly improve the fit. By adding dark matter, the effective Newtonian RC is the same, but V_N changes by the addition of the new component whose most significative contribution to the RC appears at large radii.

For the BG approach, there is no evidence of the same CT systematics. However, there is a less significative tendency with the opposite behaviour, that is, the V_N curve is too high at large radii. Hence, no significative improvement on the fits are expected if some dark matter profile is considered (at least considering the usual dark matter profiles whose density profile decreases much slower than the baryonic density).

The values of χ^2_p in Table 3 are significantly lower for the CT approach than for the BG one. This is expected, since the first has more free parameters to fit the observational RC. A reduced χ^2 analysis indicates that the BG approach fits better the observational RC, in the sense that its $\chi^2_{red,p}$ values are closer to 1.

A good fit related to χ^2_p is just a minimum requirement for the proposed model to work, it is not sufficient to show that the model is a good one. For the CT and the BG approaches, the χ^2 values are associated to the effective Newtonian RC fit. It is the fit related to χ^2 that is the physically meaningful fit.

Table 4 shows the stellar mass-to-light ratios. The expected ranges for ESO 116-G12 and ESO 287-G13 are the same stated by Gentile et al. (2004). The other galaxies expectations come from de Blok et al. (2008). We considered a factor two of uncertainty to generate the stated ranges in this table (Bell & de Jong 2001; Meidt et al. 2014), hence the lower bound is found by dividing the expected value from de Blok et al. (2008) by two, and the upper bound by multiplying it by two. The CT approach has a tendency towards higher Υ_* values, while the BG one tends towards low Υ_* values. This indicates that, by adding a dark matter halo to these approaches, the CT one may benefit from it, achieving better agreement with the expected Υ_* values, but the BG approach cannot improve and may worsen the Υ_* concordance if the presence of dark matter is considered.

Table 1. Results of the CT approach with three model parameters p_i applied to the galaxy ESO 116-G12 (see Fig. 1). This fit considers the full evaluation of the effective Newtonian data with error bars. This table also includes a comparison to the corresponding results when the same galaxy is modelled with baryonic matter and a NFW dark matter halo (NFW results from [Rodrigues et al. 2014](#)).

Galaxy	CT (3 parameters p_i)				NFW	
	χ_p^2	$\chi_{p,\text{red}}^2$	χ^2	χ_{red}^2	χ^2	χ_{red}^2
ESO 116-G12	31.60	2.63	1341.35	95.81	31.15	2.60

Table 2. The values of the p_i parameters and its errors (δp_i) for both the CT and BG approaches.

CT approach						
parameters	DDO 154	ESO 116-G12	ESO 287-G13	NGC 2403 2D	NGC 2841	NGC 3198 1D
D_1 (km/s)	303 ± 10	962 ± 30	3455 ± 87	1864 ± 13	1.155 ± 0.017	4407 ± 54
D_2 (km/s)	$6.3^{+5.7}_{-5.8}$	34 ± 22	200^{+61}_{-69}	111.2 ± 8.3	1303^{+91}_{-90}	415 ± 31
D_3 (km/s)	10.4 ± 4.4	27 ± 13	191^{+47}_{-46}	103.5 ± 6.3	928^{+78}_{-76}	280 ± 23
D_4 (km/s)	-1.1 ± 3.5	-4 ± 14	22 ± 44	26.5 ± 5.5	252 ± 55	55 ± 21
D_5 (km/s)	4.8 ± 3.0	$9.7^{+9.6}_{-9.7}$	60^{+49}_{-43}	13.4 ± 5.0	293 ± 58	21.9 ± 5.9
D_6 (km/s)	-2.1 ± 2.8	1 ± 11	12 ± 23	11.4 ± 4.5	101^{+46}_{-47}	5 ± 16
D_7 (km/s)	$1.4^{+2.7}_{-2.6}$	$2.1^{+8.0}_{-7.9}$	1 ± 43	7.1 ± 4.1	112^{+46}_{-47}	26 ± 14
D_8 (km/s)	$-0.3^{+2.3}_{-2.4}$	-1.4 ± 8.4	3^{+16}_{-15}	7.5 ± 3.8	17^{+40}_{-41}	9 ± 14
D_9 (km/s)	0.7 ± 2.0	0.6 ± 7.5	16^{+29}_{-28}	8.7 ± 3.5	33^{+39}_{-38}	11 ± 13
D_{10} (km/s)	0.2 ± 1.5	1.9 ± 6.5	-3^{+18}_{-16}	7.9 ± 3.2	60^{+28}_{-27}	-0.1 ± 11
BG approach						
parameters	DDO 154	ESO 116-G12	ESO 287-G13	NGC 2403 2D	NGC 2841	NGC 3198 1D
R (kpc)	$2.1^{+10}_{-2.1} \times 10^7$	63^{+10}_{-40}	$6.7^{+10}_{-6.7} \times 10^7$	$4.37^{+10}_{-0.10} \times 10^7$	109^{+13}_{-11}	$78.9^{+12}_{-9.3}$
r_0 (kpc)	$1.18^{+0.13}_{-0.12}$	$1.79^{+0.56}_{-0.38}$	$1.308^{+0.095}_{-0.090}$	$0.706^{+0.034}_{-0.033}$	$0.31^{+0.14}_{-0.13}$	$2.01^{+0.19}_{-0.18}$
V_0 (km/s)	$58.3^{+5.1}_{-1.8}$	146^{+40}_{-19}	$191.9^{+6.2}_{-2.3}$	$141.14^{+0.77}_{-0.76}$	$345.4^{+7.1}_{-6.8}$	$197.3^{+6.8}_{-6.3}$

Table 3. Results on χ^2 and related quantities of the CT and BG approaches. The corresponding plots are in Fig. 2 (NFW results from [Rodrigues et al. 2014](#)).

Galaxy	CT (10 parameters p_i)				BG				NFW	
	χ_p^2	$\chi_{p,\text{red}}^2$	χ^2	χ_{red}^2	χ_p^2	$\chi_{p,\text{red}}^2$	χ^2	χ_{red}^2	χ^2	χ_{red}^2
DDO 154	5.93	0.12	103.56	1.73	26.46	0.45	53.29	0.89	50.42	0.87
ESO 116-G12	8.84	1.77	200.75	14.34	63.38	1.22	12.36	0.88	31.15	2.60
ESO 287-G13	13.96	0.87	358.70	14.35	37.68	1.63	2278.12	91.12	36.33	1.58
NGC 2403 2D	239.29	0.86	9200.02	32.17	275.66	0.96	17802.40	62.25	155.59	0.55
NGC 2841	58.18	0.44	23468.80	168.84	75.79	0.55	147.06	1.06	26.52	0.19
NGC 3198 1D	22.14	0.26	2918.39	31.38	59.54	0.65	3733.66	40.14	115.67	1.27

Table 4. Results on the stellar mass-to-light ratios of the CT and BG approaches shown in comparison with the NFW profile results and the expected values from stellar population considerations. Expected values on Υ_* and NFW results are from Refs. ([Gentile et al. 2004](#); [de Blok et al. 2008](#); [Rodrigues et al. 2014](#)).

Galaxy	CT (10 parameters)		BG		NFW		Expected	
	Υ_D	Υ_B	Υ_D	Υ_B	Υ_D	Υ_B	$\langle \Upsilon_D \rangle$	$\langle \Upsilon_B \rangle$
DDO 154	4.18	-	3.24	-	1.25	-	0.2-0.6	-
ESO 116-G12	0.80	-	0.55	-	0.05	-	0.5-1.8	-
ESO 287-G13	1.16	-	0.63	-	1.69	-	0.5-1.8	-
NGC 2403 2D	2.39	0.00	0.23	2.17	0.32	0.63	0.2-0.8	0.3-1.2
NGC 2841	1.66	0.00	0.005	0.24	0.72	1.28	0.4-1.5	0.4-1.7
NGC 3198 1D	1.07	-	0.41	-	0.51	-	0.4-1.6	-

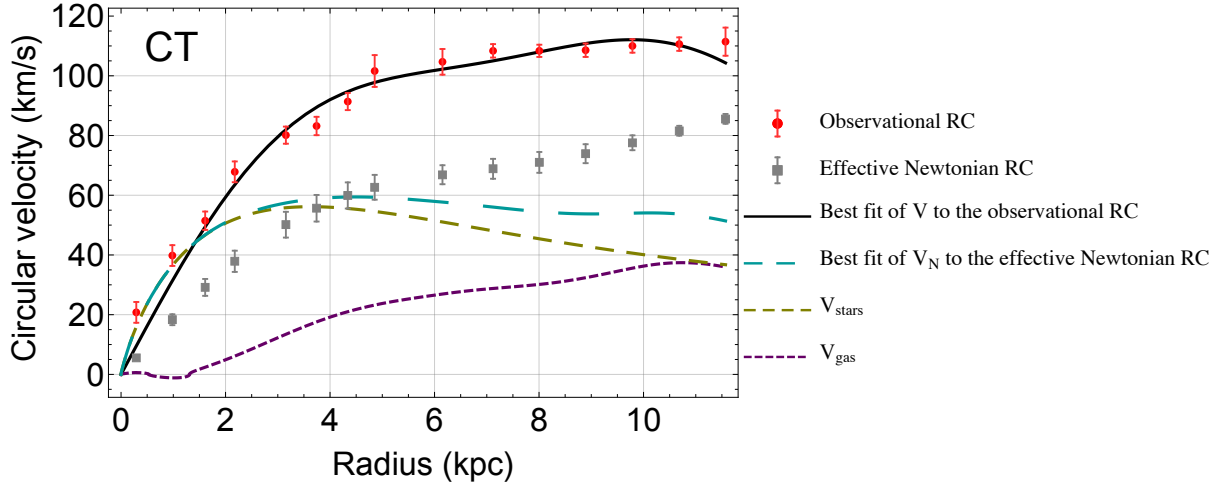


Figure 1. The RC curve analysis of ESO 116-G12 with the CT approach with three free model parameters (p_i). The fit of the total Newtonian circular velocity V_N to the effective Newtonian RC is not satisfactory ($\chi^2_{\text{red}} \gg 10$).

5 CONCLUSIONS AND DISCUSSION

There are some models that use the observational rotation curve (RC) data of galaxies to derive the matter density, while the usual route is the opposite one. This class of models have appeared in the context of pure General Relativity (GR) in four dimensional spacetime (e.g. Cooperstock & Tieu 2007; Balasin & Grumiller 2008), in higher dimensions (e.g., Coimbra-Araujo & Letelier 2007), or in GR extensions (e.g., Dey et al. 2015). Not always the corresponding papers have clearly stated that they were doing this inversion, and sometimes a simple and successful fit to the observational RC was claimed as an evidence that the proposal can model the internal dynamics of galaxies, even without dark matter. To simply mimic the form of some galaxy RCs is not sufficient (this is one of the general criticisms of Salucci & Gentile (2006)).

Here we propose a new method to properly evaluate the application of these approaches when confronted to observational galaxy data. The method only relies on data that can be commonly found in publications on galaxy RCs. Namely, it depends on the observational RC data, the stellar density profile and the gaseous density profile.

The method consists of converting the observational RC into a dataset of an effective RC that should be fitted using standard Newtonian gravity procedures. The latter dataset is called the effective Newtonian RC. This conversion is model dependent. The method can be used to consider non-Newtonian gravity models with or without some dark matter.

The method was here applied to two approaches based on GR, the CT (Cooperstock & Tieu 2007) and the BG (Balasin & Grumiller 2008) approaches. Merits and issues related to the theoretical basis of these approaches were briefly reviewed in sec. 2. Our focus was here on testing their phenomenological results, in particular since some publications related to the CT approach (e.g., Cooperstock & Tieu 2007; Carrick & Cooperstock 2012) stated that this approach is capable of achieving good results on the RC fitting of diverse galaxies without dark matter (or with a small amount

of it). The BG approach was here confronted with the astrophysical data of galaxies for the first time.

The application of the effective Newtonian RC method has shown that both the approaches have strong problems fitting galaxy RCs without dark matter (the selected sample favours the BG approach over the CT one). The method also indicates that if dark matter is considered, the BG approach cannot improve its results significantly, but the CT approach can.

For an example, we consider the case of the BG approach applied to the galaxy ESO 287-G13, Fig. 2. The BG RC could match nicely the observational RC, but is effective Newtonian RC at large radii is smaller than the contribution from the gas alone, hence this model and the data are not compatible (this can also be seen from its large χ^2_{red} value at Table 3). Moreover, by adding dark matter to the analyses, the problem is not alleviated, but it increases. By adding any dark matter halo whose most significant RC contribution is at large radii, the Newtonian curve V_N (the dashed cyan curve in the plots) will become higher at large radii. This is the opposite to what happens for the same galaxy with the CT approach. For the latter, the V_N curve is below the effective Newtonian RC at large radii.

Beyond the two GR approaches tested here, we expect that the evaluation of other models could benefit from the method that was here introduced.

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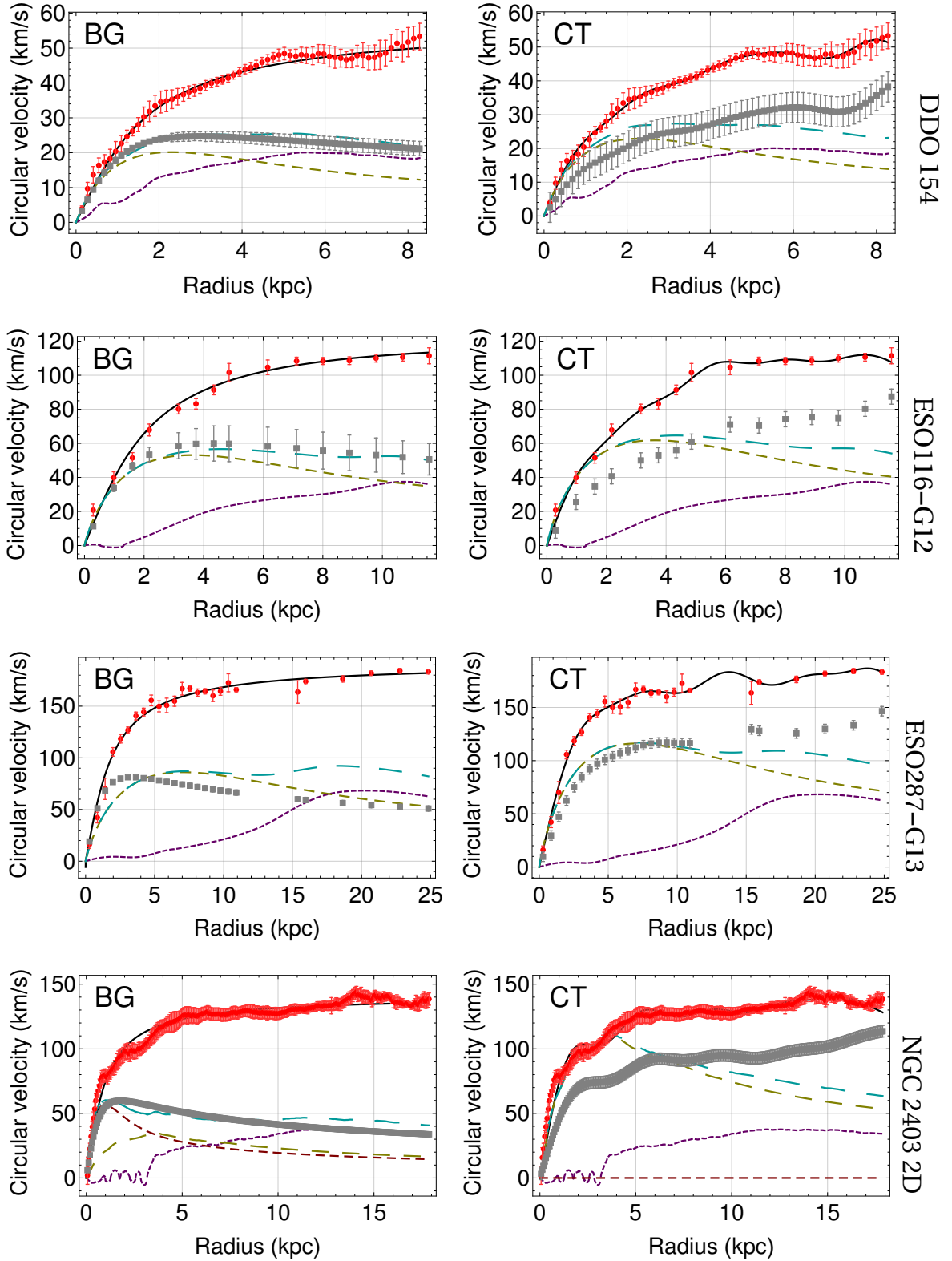


Figure 2. Rotation curves of the galaxies DDO 154, ESO 116-G12, ESO 287-G13 and NGC 2403 2D. The plots use the same conventions of Fig. 1, with the addition that the dashed dark red curve, when present, refers to the bulge circular velocity (V_{bulge}).

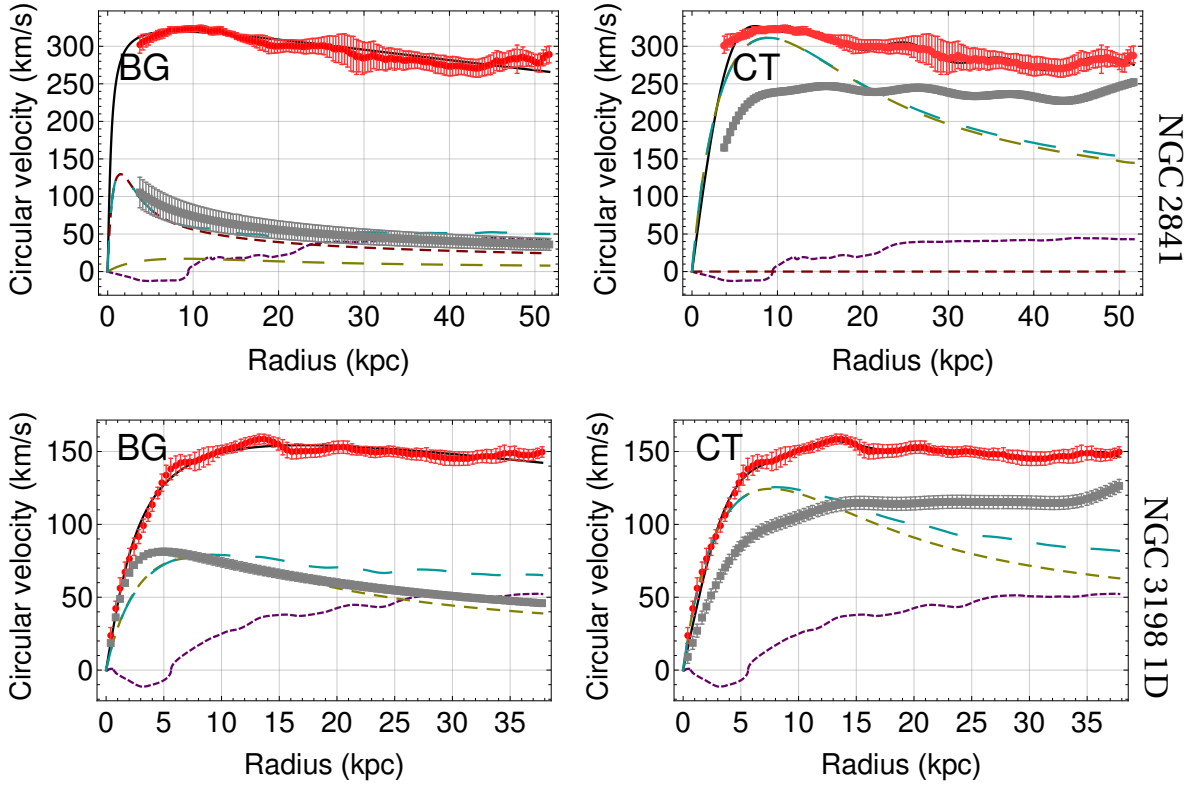


Figure 2 – *continued* Rotation curves of the galaxies NGC 2841 and NGC 3198 1D.

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